

## ‘NULL GAUGED’ WZNW THEORIES AND TODA-LIKE $\sigma$ -MODELS

C. Klimčík \*

*Theory Division, CERN,  
CH-1211 Geneva 23, Switzerland*

### Abstract

A new method of gauging of WZNW models is presented, leading to a class of exact string solutions with a target space metric of Minkowskian signature. The corresponding models may be interpreted as  $\sigma$ -model analogues of the Toda field theories.

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\* e-mail: klimcik@surya20.cern.ch

## 1. Introduction

Because of possible implications of string theory for gravitational physics it is important to study string solutions with realistic signature  $(-, +, +, \dots)$ . In trying to understand issues of singularities and short distance structure one is mostly interested not just in solutions of the leading-order low-energy string effective equations but in the ones that are exact in  $\alpha'$  and/or which have an explicit conformal field theory interpretation. While the leading-order solutions are numerous, very few solutions of the second type are known. In this contribution, based on the original work [1], we present some new exact solutions which correspond to gauged WZNW models and thus should have a direct conformal field theory interpretation. The ‘null’ gauging means the gauging of WZNW models [2][3][4][5] based on non-compact [6][7][8] groups with the generators of the gauged subgroup being ‘null’ (having zero Killing scalar products). The gauged subgroup will be thus chosen to be solvable (but need not be nilpotent in general). The resulting sigma models will belong to the following class<sup>1</sup>

$$S = \frac{1}{\pi\alpha'} \int d^2z [\partial x^i \bar{\partial} x_i + F(x) \partial u \bar{\partial} v] + \frac{1}{4\pi} \int d^2z \sqrt{g^{(2)}} R^{(2)} \phi(x) , \quad (0)$$

where the two functions  $F$  and  $\phi$  will be explicitly determined (in Section 2) by gauging of the nilpotent subgroups in WZNW models for rank  $n$  maximally non-compact groups.  $x^i$  ( $i = 1, \dots, n$ ) will be the linear combinations of the coordinates  $r^i \equiv r^{\alpha_i}$  corresponding to the simple roots  $\alpha_i$ ,

$$\partial x^i \bar{\partial} x_i = C_{ij} \partial r^i \bar{\partial} r^j , \quad \alpha_i \cdot x = K_{ij} r^j , \quad K_{ij} \equiv K_{\alpha_i \alpha_j} = \frac{2\alpha_i \cdot \alpha_j}{|\alpha_j|^2} = \frac{1}{2} |\alpha_i|^2 C_{ij} ,$$

where  $K_{ij}$  is the  $n \times n$  Cartan matrix. We shall find that

$$F = \frac{1}{\sum_{i=1}^n \epsilon_i e^{\alpha_i \cdot x}} , \quad \phi = \frac{1}{2} \sum_{s=1}^m \alpha_s \cdot x - \frac{1}{2} \ln \left( \sum_{i=1}^n \epsilon_i e^{\alpha_i \cdot x} \right) = \rho \cdot x + \frac{1}{2} \ln F ,$$

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<sup>1</sup> The so-called F-model considered in [9], where the conditions of (all order in  $\alpha'$ ) conformal invariance were given for generic  $F$  and  $\phi$ .

where the constants  $\epsilon_i$  can be chosen to be 0 or  $\pm 1$  and  $\rho = \frac{1}{2} \sum_{s=1}^m \alpha_s$  is half of the sum of all positive roots.<sup>2</sup>

In Section 3 we discuss an apparent similarity to the Toda model (in particular, we shall find a direct relation between the solutions of the classical equations of motion). Some concluding remarks will be also made, concerning the dual (in  $u + v$  direction) version of the above  $\sigma$ -model.

## 2. Null gauging of WZNW models

### 2.1. General scheme

The indefinite signature of the Killing form for non-compact algebras implies that there is a number of ‘null’ generators  $T_n = N_n$  which have zero scalar products,  $\text{Tr} (N_n N_m) = 0$ . A subalgebra generated by such generators is thus solvable (but may not be nilpotent).<sup>3</sup> In this case one can consider a left-right asymmetric gauging since the anomaly cancellation condition [15] ( $\text{Tr} T_L^2 = \text{Tr} T_R^2$ ) is obviously satisfied.

Consider the action of the ‘null’ gauged WZNW model

$$S = -kI(g) - \frac{k}{\pi} \int d^2z \text{Tr} (-A \bar{\partial} gg^{-1} + \bar{A} g^{-1} \partial g + g^{-1} A g \bar{A}) = -kI(h^{-1} g \bar{h}), \quad (1)$$

where

$$I \equiv \frac{1}{2\pi} \int d^2z \text{Tr} (\partial g^{-1} \bar{\partial} g) + \frac{i}{12\pi} \int d^3z \text{Tr} (g^{-1} dg)^3.$$

and<sup>4</sup>

$$A = h \partial h^{-1}, \quad \bar{A} = \bar{h} \bar{\partial} \bar{h}^{-1}, \quad h \in H_+, \bar{h} \in H_-,$$

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<sup>2</sup> If  $\alpha_1$  is a simple root corresponding to the generators  $E_{\pm\alpha_1}$  which are left ungauged (the remaining  $m - 1$  positive (negative) roots correspond to the generators of a left (right) nilpotent subgroup that was gauged) then  $\epsilon_1 = 1$  ( $m = \frac{1}{2}(d - n)$ ,  $n = \text{rank } G$ ,  $d = \dim G$ ).

<sup>3</sup> An example of a ‘null’ generator in the Lorentz group case is a sum of a spatial rotation with a boost. Note that a nilpotent ( $N^2 = 0$ ) generator is null but, in general, a null generator need not be nilpotent. Gauging of subgroups generated by nilpotent generators was previously discussed in [8][10][11][12][13][14].

<sup>4</sup>  $A$  and  $\bar{A}$  should be considered as chiral projections of the two independent vector fields

Here  $H_+, H_-$  are two different subgroups of  $G$  generated by null generators ( $\text{Tr}(N_n N_m) = 0$ ). The action (1) is local and manifestly gauge invariant under the  $H^\pm$ -gauge transformation

$$g \rightarrow w^{-1}g\bar{w}, \quad A \rightarrow w^{-1}(A + \partial)w, \quad \bar{A} \rightarrow \bar{w}^{-1}(\bar{A} + \bar{\partial})\bar{w}, \quad h \rightarrow w^{-1}h, \quad \bar{h} \rightarrow \bar{w}^{-1}\bar{h}, \quad (2)$$

Assuming that the corresponding quantum theory is regularised in the ‘left-right decoupled’ way (so that the local counterterm  $\text{Tr}(A\bar{A})$  does not appear) the only non-trivial renormalisation that can occur at the quantum level is the shift of the overall coefficient  $k \rightarrow k - \frac{1}{2}c_G$ <sup>5</sup> in front of the action (1). As a result, the couplings of the sigma model obtained by integrating out the gauge fields  $A, \bar{A}$  should not receive non-trivial  $k^{-1}$ -corrections, i.e. they should represent an exact solution of the sigma model conformal invariance equations. The central charge of the resulting gauged model will be equal to the central charge of the original WZNW model minus the dimension of the gauged subgroup. Depending on a choice of the null subgroups  $H_+$  and  $H_-$  the trace of the product of their generators  $\text{Tr}(N\bar{N})$  and hence  $\text{Tr}(A\bar{A})$  may or may not vanish so that (1), in general, is different from the action of the vectorially gauged WZNW model

$$\begin{aligned} I_v(g, A) &= I(h^{-1}g\bar{h}) - I(h^{-1}\bar{h}) = I(g) + \frac{1}{\pi} \int d^2z \text{Tr}(-A\bar{\partial}gg^{-1} + \bar{A}g^{-1}\partial g \\ &\quad + g^{-1}Ag\bar{A} - A\bar{A}) \equiv I_0(g, A) - \frac{1}{\pi} \int d^2z \text{Tr}(A\bar{A}), \end{aligned} \quad (3)$$

## 2.2. Gauging of nilpotent subgroups in Gauss decomposition parametrisation

A particular case of such gauging (when the null subgroups are the nilpotent subgroups corresponding to the step generators in the Gauss decomposition) was considered previously [8][10](see also [11][12]) in the context of Hamiltonian reduction [16][17]

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<sup>5</sup>  $c_G$  is the value of the quadratic Casimir operator in adjoint representation. The negative sign of the shift is due to our choice of the ‘unphysical’ sign in the action (1) as usual in the non-compact case.

of WZNW theories related to Toda models.<sup>6</sup> The approach based on gauging of any subgroup with null generators is more general since, in principle, we do not need to use the Gauss decomposition (which does not always exist for the real groups we are to consider to get a real WZNW action). The gauging based on the Gauss decomposition directly applies only to the groups with the algebras that are the ‘maximally non-compact’ real forms of the classical Lie algebras (real linear spans of the Cartan-Weyl basis), i.e.  $sl(n+1, R)$ ,  $so(n, n+1)$ ,  $sp(2n, R)$ ,  $so(n, n)$ . The corresponding WZNW models can be considered as natural generalisations of the  $SL(2, R)$  WZNW model. For these groups there exists a real group-valued Gauss decomposition

$$g = g_+ g_0 g_- , \quad g_+ = \exp\left(\sum_{\Phi_+} u^\alpha E_\alpha\right) , \quad g_- = \exp\left(\sum_{\Phi_+} v^\alpha E_{-\alpha}\right) , \quad (4)$$

$$g_0 = \exp\left(\sum_{\Delta} r^\alpha H_\alpha\right) = \exp\left(\sum_{i=1}^n x^i H_i\right) .$$

Here  $\Phi_+$  and  $\Delta$  are the sets of the positive and simple roots of a complex algebra with the Cartan-Weyl basis consisting of the step operators  $E_\alpha$ ,  $E_{-\alpha}$ ,  $\alpha \in \Phi_+$  and  $n$  ( $= \text{rank } G$ ) Cartan subalgebra generators  $H_\alpha$ ,  $\alpha \in \Delta$ .<sup>7</sup> We shall use the following standard relations (we shall assume that a long root has  $|\alpha|^2 = 2$ ) [19][20]

$$[H_\alpha, E_\beta] = K_{\beta\alpha} E_\beta \quad (\alpha \in \Delta, \beta \in \Phi) ,$$

$$[E_\alpha, E_{-\alpha}] = H_\alpha \quad (\alpha \in \Delta) , \quad [E_\alpha, E_\beta] = N_{\alpha\beta} E_{\alpha+\beta} ,$$

$$\text{Tr } (H_\alpha H_\beta) \equiv C_{\alpha\beta} = \frac{2}{|\alpha|^2} K_{\alpha\beta} , \quad K_{\alpha\beta} = \frac{2\alpha \cdot \beta}{|\beta|^2} , \quad \text{Tr } (H_i H_j) = \delta_{ij} ,$$

$$H_\alpha = \tilde{\alpha}^i H_i , \quad , \quad x^i = \sum_{\alpha \in \Delta} \tilde{\alpha}^i r^\alpha , \quad K_{\alpha\beta} r^\beta = \alpha \cdot x , \quad \tilde{\alpha}^i \equiv \frac{2}{|\alpha|^2} \alpha^i ,$$

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<sup>6</sup> The WZNW model in the Gauss decomposition parametrisation was considered in [18]. The standard (vector or axial) gauging in the Gauss decomposition was also discussed in [13].

<sup>7</sup> In what follows we shall assume that there is always a sum over repeated upper and lower indices. We shall also use  $r^\alpha$  with understanding that  $r^\alpha \neq 0$  only if  $\alpha$  is a simple root.

$$\mathrm{Tr} (E_\alpha E_{-\beta}) = \frac{2}{|\alpha|^2} \delta_{\alpha\beta} , \quad \mathrm{Tr} (E_\alpha H_\beta) = 0 , \quad (5)$$

where  $\alpha^i$  ( $i = 1, \dots, n$ ) are the components of the positive root vectors. It is clear that  $E_\alpha$  and  $E_{-\alpha}$  form sets of null generators so that some of the corresponding symmetries can be gauged according to (1). For example, we may take  $w$ ,  $A$  and  $h$  in (2) to belong to the one-dimensional subgroup generated by some  $E_\gamma$  and  $\bar{w}, \bar{A}$  and  $\bar{h}$  – to the subgroup generated by  $E_{-\gamma'}$  where  $\gamma$  and  $\gamma'$  are positive roots which may not necessarily be the same.

If one gauges the full left and right nilpotent subgroups  $G_+$  and  $G_-$  ( $\dim G_\pm = \frac{1}{2}(d-n) = m$ ,  $\dim G = d$ ) generated by all generators  $E_\alpha$  and  $E_{-\alpha}$  [8] one is left with the action for  $n$  decoupled scalars  $r^\alpha$  or  $x^i$  which represent the free part of the Toda model action. Being interested in finding non-trivial conformal sigma models describing string solutions we are to consider the more general case of ‘partial’ gauging when only some subgroups  $H_+$  and  $H_-$  of  $G_+$  and  $G_-$  are gauged.  $r^\alpha$  should correspond to spatial directions ( $C_{\alpha\beta}$  in (5) is positive definite). Since the Killing form of the maximally non-compact groups has  $m = \frac{1}{2}(d-n)$  time-like directions, to get a physical signature of the resulting space-time we need to gauge away all but *one* pair of coordinates  $u, v$  in (4). Therefore the gauge groups  $H_\pm$  should have dimension  $m-1 = \frac{1}{2}(d-n)-1$ , i.e.

$$\dim H_\pm = \dim G_\pm - 1 .$$

As we shall see below (in Sect.2.3) the ungauged generator(s) of  $G_\pm$  must be a simple root. Moreover, to get the physical value  $D = 4$  of the target space dimension we need to start with the rank 2 groups  $G$  ( $D = n+2 = 4$ ).

Let us first consider the most general case when  $w$ ,  $A$  and  $h$  in (2) correspond to the subgroup  $H_+ \subset G_+$  generated by some  $s \leq m$  linear combinations  $\mathcal{E}_p = \lambda_p^\alpha E_\alpha$  of the generators ( $E_\alpha$ ,  $\alpha \in \Phi_+$ ) of  $G_+$  and  $\bar{w}, \bar{A}$  and  $\bar{h}$  – to the subgroup  $H_- \subset G_-$  generated by some  $s' = s$  linear combinations  $\bar{\mathcal{E}}_q = \bar{\lambda}_q^\alpha E_{-\alpha}$ . Then it is straightforward to write down

the resulting expression for the action (1) using the Polyakov-Wiegmann formula and (5) (i.e.  $I(g_+) = I(g_-) = 0$ , etc.)

$$\begin{aligned} I_n &= I(h^{-1}g\bar{h}) = I(g_0) + \frac{1}{\pi} \int d^2z \operatorname{Tr} [g_0^{-1}g_+^{-1}h\partial(h^{-1}g_+)g_0g_-\bar{h}\bar{\partial}(\bar{h}^{-1}g_-^{-1})] \\ &= I(g_0) + \frac{1}{\pi} \int d^2z \operatorname{Tr} [g_0^{-1}(A + g_+^{-1}\partial g_+)g_0(\bar{A} - \bar{\partial}g_-g_-^{-1})] . \end{aligned} \quad (6)$$

Setting

$$\begin{aligned} A &= \mathcal{E}_p B^p = \lambda_p^\alpha B^p E_\alpha , \quad \bar{A} = \bar{\mathcal{E}}_q \bar{B}^q = \bar{\lambda}_q^\alpha \bar{B}^q E_{-\alpha} , \\ g_+^{-1}\partial g_+ &\equiv J_u = U_\alpha^\beta(u)\partial u^\alpha E_\beta , \quad \bar{\partial}g_-g_-^{-1} \equiv \bar{J}_v = \bar{J}_v^\alpha E_{-\alpha} = V_\alpha^\beta(v)\bar{\partial}v^\alpha E_{-\beta} , \end{aligned} \quad (7)$$

we get

$$\begin{aligned} S_n &= \frac{k}{\pi} \int d^2z \left[ \frac{1}{2} C_{\alpha\beta} \partial r^\alpha \bar{\partial} r^\beta + M_{\alpha\beta} (J_u^\alpha + \lambda_p^\alpha B^p)(\bar{J}_v^\beta - \bar{\lambda}_q^\beta \bar{B}^q) \right] \\ &= \frac{k}{2\pi} \int d^2z \left[ \partial x^i \bar{\partial} x_i + 2M_{\alpha\beta} (J_u^\alpha + \lambda_p^\alpha B^p)(\bar{J}_v^\beta - \bar{\lambda}_q^\beta \bar{B}^q) \right] , \end{aligned} \quad (8)$$

where

$$M_{\alpha\beta} = \operatorname{Tr} (g_0^{-1} E_\alpha g_0 E_{-\beta}) = f_\alpha(r) \delta_{\alpha\beta} , \quad f_\alpha(r) \equiv \frac{2}{|\alpha|^2} e^{-K_{\alpha\beta} r^\beta} = \frac{2}{|\alpha|^2} e^{-\alpha \cdot x} .$$

The sums over  $\alpha, \beta$  run over positive roots ( $r^\alpha \neq 0$  for simple roots only). It is clear that when  $H_\pm = G_\pm$ , i.e. when  $\lambda_p^\alpha$  and  $\bar{\lambda}_q^\beta$  are non-degenerate we can eliminate  $J_u$  and  $\bar{J}_v$  from the action by redefining the gauge fields  $B, \bar{B}$ . One is then left with the free action for  $r^\alpha$  plus the dilaton term  $\phi = \phi_0 + \frac{1}{2} \sum_\alpha K_{\alpha\beta} r^\beta$  originating from the  $B, \bar{B}$  -determinant.

Integrating out  $B^p$  and  $\bar{B}^q$  in (8) we get

$$\begin{aligned} S_n &= \frac{k}{\pi} \int d^2z \left[ \frac{1}{2} C_{\alpha\beta} \partial r^\alpha \bar{\partial} r^\beta + \mathcal{M}_{\alpha\beta}(r) U_\gamma^\alpha(u) V_\delta^\beta(v) \partial u^\gamma \bar{\partial} v^\delta \right] \\ &\quad - \frac{1}{8\pi} \int d^2z \sqrt{g^{(2)}} R^{(2)} \ln \det M_{pq}(r) , \end{aligned} \quad (9)$$

$$M_{pq}(r) \equiv M_{\alpha\beta} \lambda_p^\alpha \bar{\lambda}_q^\beta = \sum_\alpha f_\alpha(r) \lambda_p^\alpha \bar{\lambda}_q^\alpha ,$$

$$\mathcal{M}_{\alpha\beta}(r) = M_{\alpha\beta} - M^{-1pq} \bar{\lambda}_p^\gamma \lambda_q^\delta M_{\alpha\gamma} M_{\beta\delta} = f_\alpha \delta_{\alpha\beta} - f_\alpha f_\beta M^{-1pq} \bar{\lambda}_{\alpha p} \lambda_{\beta q} . \quad (10)$$

### 2.3. Models with one time-like coordinate

Let us now turn to the most interesting case when the dimensions of the gauge groups  $H_{\pm}$  are equal to  $\dim G_{\pm} - 1 = m - 1$  so that only one time-like coordinate appears in the resulting sigma model action. Let  $E_{\alpha_1}$  and  $E_{-\alpha_1}$  denote the generators of  $G_+$  and  $G_-$  which remain ungauged, i.e. which do not belong to  $H_+$  and  $H_-$ . Since  $H_+$  must be a subgroup,  $E_{\alpha_1}$  cannot appear in the commutators of the generators of  $H_+$ . According to (5) this is possible only if  $\alpha_1$  is a simple root, i.e. if it cannot be represented as a sum of two other positive roots. In fact, if we use the indices  $i, j$  to denote the simple roots  $\alpha = \alpha_i$  ( $i = (1, s) = 1, 2, \dots, n$ ) and indices  $a, b$  to denote the remaining positive roots  $\alpha_a$  ( $a = n + 1, \dots, m$ ) the commutators of the corresponding step operators are given by

$$[E_i, E_j] \sim E_a \quad (\alpha_a = \alpha_i + \alpha_j) ,$$

$$[E_i, E_a] \sim E_b \quad (\alpha_b = \alpha_i + \alpha_a) , \quad [E_a, E_b] \sim E_c \quad (\alpha_c = \alpha_a + \alpha_b) .$$

It is clear that one can also use linear combinations  $E'_s = E_s + \lambda_s E_1$  ( $s = 2, \dots, n$ ) as the ‘simple’ part of the generators of  $H_+$  but one cannot mix the non-simple generators  $E_a$  with  $E_1$ .

Let  $u^{\alpha_1} \equiv \frac{1}{\sqrt{2}}u$ ,  $v^{\alpha_1} \equiv \frac{1}{\sqrt{2}}v$ ; the remaining coordinates  $u^{\sigma}, v^{\sigma}$  ( $\sigma = (s, a)$ ) will be used to denote all ‘gauged’  $m - 1$  positive roots) are transforming under the gauge group (with the leading-order term being just a shift) so that we can set them to zero as a gauge. In this gauge  $J_u = \frac{1}{\sqrt{2}}\partial u E_{\alpha_1}$ ,  $\bar{J}_v = \frac{1}{\sqrt{2}}\bar{\partial}v E_{-\alpha_1}$  and the sigma model action (9) takes the form ( $p, q = 1, \dots, m - 1$ )<sup>8</sup>

$$S_n = \frac{k}{2\pi} \int d^2z [\partial x^i \bar{\partial} x_i + F(x) \partial u \bar{\partial} v] + \frac{1}{4\pi} \int d^2z \sqrt{g^{(2)}} R^{(2)} \phi(x) , \quad (11)$$

$$F(x) = f_{\alpha_1} - f_{\alpha_1}^2 M^{-1pq} \bar{\lambda}_p^{\alpha_1} \lambda_q^{\alpha_1} , \quad \phi(x) = -\frac{1}{2} \ln \det M_{pq} ,$$

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<sup>8</sup> For notational convenience (to get rid of an extra factor of 2 in front of the  $F(x)\partial u \bar{\partial} v$  term) we have redefined  $u$  and  $v$  by the factor of  $1/\sqrt{2}$  as compared to (9).

$$M_{pq}(x) = \sum_{\alpha} f_{\alpha}(x) \lambda_p^{\alpha} \bar{\lambda}_q^{\alpha} .$$

The non-trivial elements of the ‘mixing’ matrix  $\lambda_p^{\alpha}$  correspond to a possibility of changing the generators of  $H_+$  by adding  $\lambda_p^{\alpha_1} E_{\alpha_1}$ . Without loss of generality the non-vanishing components of  $\lambda_p^{\alpha}$  can be taken to be:  $\lambda_p^{\sigma} = \delta_p^{\sigma}$ ,  $\lambda_p^{\alpha_1} = \lambda_s \delta_{ps}$  and similarly for  $\bar{\lambda}_q^{\alpha}$  (according to the remark above, only simple roots can be mixed with  $E_{\alpha_1}$ ). Then ( $s, t = 2, \dots, n$ ;  $a, b = n+1, \dots, m$ )

$$M_{pq} = \begin{pmatrix} M_{st} & 0 \\ 0 & M_{ab} \end{pmatrix}, \quad M_{st}(r) = f_s \delta_{st} + f_1 \lambda_s \bar{\lambda}_t, \quad M_{ab}(x) = f_a(x) \delta_{ab}, \quad (12)$$

$$f_h(x) \equiv f_{\alpha_h} = \frac{2}{|\alpha_h|^2} e^{-\alpha_h \cdot x}, \quad h = (1, s, a) = 1, 2, \dots, m, \quad m = \frac{1}{2}(d - n). \quad (13)$$

If we introduce  $\lambda_1 = \bar{\lambda}_1 = 1$  in order to make the formulas look symmetric with respect to all simple roots, we find

$$M_{st}^{-1} = f_s^{-1} \delta_{st} - \frac{f_s^{-1} f_t^{-1} \lambda_s \bar{\lambda}_t}{\sum_{i=1}^n f_i^{-1} \lambda_i \bar{\lambda}_i}, \quad \det M_{pq} = \left( \prod_{h=1}^m f_h \right) \left( \sum_{i=1}^n f_i^{-1} \lambda_i \bar{\lambda}_i \right). \quad (14)$$

As a result,

$$F = f_1 - f_1^2 M^{-1st} \bar{\lambda}_s \lambda_t = \frac{1}{\sum_{i=1}^n f_i^{-1} \lambda_i \bar{\lambda}_i},$$

$$F = \left( \sum_{i=1}^n \epsilon_i e^{\alpha_i \cdot x} \right)^{-1}, \quad \epsilon_i \equiv \frac{1}{2} |\alpha_i|^2 \lambda_i \bar{\lambda}_i, \quad (15)$$

$$\phi = -\frac{1}{2} \sum_{h=1}^m \ln f_h - \frac{1}{2} \ln \sum_{i=1}^n f_i^{-1} \lambda_i \bar{\lambda}_i,$$

$$\phi = \phi_0 + \rho \cdot x - \frac{1}{2} \ln \sum_{i=1}^n \epsilon_i e^{\alpha_i \cdot x}, \quad \rho = \frac{1}{2} \sum_{h=1}^m \alpha_h. \quad (16)$$

We have thus found the sigma model action (11),(15),(16), i.e.<sup>9</sup>

$$S_n = \frac{\kappa}{2\pi} \int d^2 z [\partial x^i \bar{\partial} x_i + \left( \sum_{i=1}^n \epsilon_i e^{\alpha_i \cdot x} \right)^{-1} \partial u \bar{\partial} v]$$

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<sup>9</sup> We absorb the prefactors  $2/|\alpha_i|^2$  of the exponential terms into a rescaling of  $u, v$  and  $\epsilon_i$ . We also consider (17) as an effective action, including the quantum shift of  $k$ ,  $\kappa \equiv k - \frac{1}{2} c_G$ .

$$+\frac{1}{4\pi} \int d^2z \sqrt{g^{(2)}} R^{(2)} \left( \rho \cdot x - \frac{1}{2} \ln \sum_{i=1}^n \epsilon_i e^{\alpha_i \cdot x} \right) , \quad (17)$$

with  $\alpha_i$  being the simple roots and  $\rho$  being half the sum of all positive roots. The values of the parameters  $\epsilon_i = 0, +1$  or  $-1$  represent inequivalent gaugings of the original WZNW model or different conformal sigma models.<sup>10</sup> Non-trivial models (not equivalent to direct products of  $SL(2, R)$  WZNW with free scalars) are found for non-vanishing values of the ‘mixing’ parameters  $\epsilon_2, \dots, \epsilon_n$ .

The metric of the corresponding  $D = n + 2$  dimensional target space-time has two null Killing symmetries (in fact, the full  $2d$  Poincare invariance in the  $u, v$  plane,  $u' = \rho u + a, v' = \rho^{-1}v + b$ ). The non-trivial ( $uv$ ) components of the metric and the antisymmetric tensor and the non-linear part of the dilaton are all expressed in terms of a single function  $F(x)$  (15), which is the inverse of the sum of the exponentials of the spatial Cartan coordinates  $x^i$ . The metric is non-singular if all  $\epsilon_i$  have the same sign.

### 3. Relation to the Toda models

To demonstrate the equivalence to the Toda model at the classical level let us consider the classical equations for the model (0) (on a flat  $2d$  background)

$$\partial\bar{\partial}x_i - \frac{1}{2}\partial_i F\partial u\bar{\partial}v = 0 , \quad \partial(F\bar{\partial}v) = 0 , \quad \bar{\partial}(F\partial u) = 0 . \quad (18)$$

The model thus has two chiral currents. Integrating the last two equations and substituting the solutions in the first one we get

$$\partial\bar{\partial}x_i + \frac{1}{2}\chi\partial_i F^{-1} = 0 , \quad F\bar{\partial}v = \nu(\bar{z}) , \quad F\partial u = \mu(z) , \quad \chi \equiv \nu(\bar{z})\mu(z) . \quad (19)$$

Since  $\nu, \mu$  are chiral and  $\chi$  has a factorised form they can be made constant by the conformal transformations of  $z$  and  $\bar{z}$  (the sigma model (0) is always conformally invariant at the

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<sup>10</sup> Inequivalent solutions corresponding to different possible choices of an ungauged simple root  $\alpha_1$  are easily included by assuming that  $\epsilon_1$  can also take values 0 and  $-1$  but at least one of  $\epsilon_i$  is non-vanishing. In general,  $\epsilon_i$  taking arbitrary real values represent moduli of the solutions.

classical level). Equivalently, this can be considered as a gauge choice (alternative to the light cone gauge), i.e.  $u = a(\tau + \sigma)$ ,  $v = b(\tau - \sigma)$ , for the conformal symmetry. The equation for  $x^i$  can be derived from the action

$$S = \frac{1}{\pi\alpha'} \int d^2z [\partial x^i \bar{\partial} x_i - \chi F^{-1}(x)] . \quad (20)$$

With  $F$  given by (15) (i.e.  $F^{-1} = T = \sum_{i=1}^n \epsilon_i e^{\alpha_i \cdot x}$ ) and in the conformal gauge with constant  $\chi$  the equation for  $x^i$  (19) and the action (20) are exactly those of the Toda model. This observation is, of course, related to the derivation of the Toda model from constrained WZNW model in [8].

As a consequence, the equations of the classical string propagation (including the constraints) on the backgrounds (17) discussed in the present paper are exactly integrable since their solutions can be directly expressed in terms of the Toda model solutions.

Another interesting property of the model (0) is that the dual model obtained by the standard abelian duality transformation in the  $w = \frac{1}{2}(u + v)$  direction has a covariantly constant null Killing vector, i.e. has a ‘plane wave’ - type structure (with the metric and dilaton depending only on transverse coordinates but not on a light-cone coordinate). This is a consequence of the fact that the  $uv$ -component of the metric in (0) is equal to the corresponding component of the antisymmetric tensor (what is also the reason for the existence of the two chiral currents). Following the standard steps [21][22] of gauging the symmetry  $u' = u + a$ ,  $v' = v + a$ , i.e. adding the gauge field strength term with a Lagrange multiplier  $\tilde{u}$  and integrating out the gauge field, we find for the dual model ( $t = \frac{1}{2}(u - v) \equiv \tilde{v}$ )

$$\tilde{S} = \frac{1}{\pi\alpha'} \int d^2z [\partial x^i \bar{\partial} x_i + \tilde{F}(x) \partial \tilde{u} \bar{\partial} \tilde{u} - 2\partial \tilde{v} \bar{\partial} \tilde{u}] + \frac{1}{4\pi} \int d^2z \sqrt{g^{(2)}} R^{(2)} \tilde{\phi}(x) , \quad (21)$$

$$\tilde{F} = F^{-1}(x) , \quad \tilde{\phi} = \phi(x) - \frac{1}{2} \ln F(x) . \quad (22)$$

The dual space-time metric has one covariantly constant null Killing vector while the anti-symmetric tensor vanishes. Since the ‘transverse’ part of the metric is flat, the conditions

of the Weyl invariance of the model (21) are given (to all orders in  $\alpha'$ ) by the ‘one-loop’ conditions<sup>11</sup>

$$-\partial^i \partial_i \tilde{F} + 2\partial_i \tilde{\phi} \partial^i \tilde{F} = 0 , \quad \partial_i \partial_j \tilde{\phi} = 0 , \quad (23)$$

$$\frac{26 - D}{6\alpha'} = -\frac{1}{2} \partial_i \partial^i \tilde{\phi} + \partial_i \tilde{\phi} \partial^i \tilde{\phi} .$$

The dual data to (15) obviously solve those equations, thus providing another example of exact solutions related by the standard (leading-order) duality (cf. [30]). The classical equations of motion of the two dual models are, of course, also equivalent and can be represented in the Toda-like form (19).

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<sup>11</sup> See, e.g., [23] for a general discussion. Since the dilaton is assumed to depend on the transverse coordinates, this model is a generalisation of the original ‘plane-wave’ models considered in [24][25][26][27]. Some special cases, in particular, the  $D = 3$  case of such model – the duality rotation of the  $SL(2, R)$  WZNW model – were already discussed (in connection with extremal black strings) in [28][29].

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